Words of Wisdom from Terence Tao

Solving mathematical problems

Chaque vérité que je trouvois étant une règle qui me servoit après à en trouver d'autres [Each truth that I discovered became a rule which then served to discover other truths]. (René Descartes, "Discours de la Méthode")

Problem solving, from homework problems to unsolved problems, is certainly an important aspect of mathematics, though definitely not the only one. Later in your research career, you will find that problems are mainly solved by knowledge (of your own field and of other fields), experience, patience and hard work; but for the type of problems one sees in school, college or in mathematics competitions one needs a slightly different set of problem solving skills. I do have a book on how to solve mathematical problems at this level; in particular, the first chapter discusses general problem-solving strategies. There are of course several other problem-solving books, such as Polya's classic "How to solve it", which I myself learnt from while competing at the Mathematics Olympiads.

Solving homework problems is an essential component of *really* learning a mathematical subject – it shows that you can "walk the walk" and not just "talk the talk", and in particular identifies any specific weaknesses you have with the material. It's worth persisting in trying to understand how to do these problems, and not just for the immediate goal of getting a good grade; if you have a difficulty with the homework which is not resolved, it is likely to cause you further difficulties later in the course, or in subsequent courses.

I find that "playing" with a problem, even after you have solved it, is very helpful for understanding the underlying mechanism of the solution better. For instance, one can try removing some hypotheses, or trying to prove a stronger conclusion. See "ask yourself dumb questions".

It's also best to keep in mind that obtaining a solution is only the short-term goal of solving a mathematical problem. The long-term goal is to increase your understanding of a subject. A good rule of thumb is that if you cannot adequately explain the solution of a problem to a classmate, then you haven't really understood the solution yourself, and you may need to think about the problem more (for instance, by covering up the solution and trying it again). For related reasons, one should value partial progress on a problem as being a stepping stone to a complete solution (and also as an important way to deepen one's understanding of the subject).

Dear Dr. Tao,I am an undergraduate studying math.While trying to solve problems from my text books (like Stein's Complex Analysis), I notice that very often I cannot solve the hardest problems from them. Since research is about hard problems, does that mean I don't have what it takes to be a mathematician?

Terence Tao

As you are still several years away from having to attack research-level mathematics problems, your *current* skill in solving such problems is not particularly relevant (much as the calculus-solving skill of, say, a seventh-grader, has much bearing on how good that seventh-grader will be at calculus when he or she encounters it at the college level). The more important consideration is the extent to which your problem-solving skills are *improving* over time. For instance, if after failing to solve a problem, you receive the solution and study it carefully, you may discover an insight or problem-solving technique that eluded you before, and will now be able to solve similar problems that were previously out of reach. One should also bear in mind that being able to partially solve a problem (e.g. to expand out the definitions, solve some special cases, and isolate key difficulties) is also a very important measure of progress (see this previous post of mine on this topic), as is the practice of constantly asking yourself "dumb" questions in the subject (as discussed in this post). One should also not focus on the most difficult questions, but rather on those just outside your current range.

Ask yourself dumb questions – and answer them!

Don't just read it; fight it! Ask your own questions, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? What about the degenerate cases? Where does the proof use the hypothesis? (Paul Halmos, "I want to be a mathematician")

When you learn mathematics, whether in books or in lectures, you generally only see the end product – very polished, clever and elegant presentations of a mathematical topic.

However, the process of discovering *new* mathematics is much messier, full of the pursuit of directions which were naïve, fruitless or uninteresting.

While it is tempting to just ignore all these "failed" lines of inquiry, actually they turn out to be essential to one's deeper understanding of a topic, and (via the process of elimination) finally zeroing in on the correct way to proceed.

So one should be unafraid to ask "stupid" questions, challenging conventional wisdom on a subject; the answers to these questions will occasionally lead to a surprising conclusion, but more often will simply tell you why the conventional wisdom is there in the first place, which is well worth knowing.

It's also acceptable, when listening to a seminar, to ask "dumb" but constructive questions to help clarify some basic issue in the talk (e.g. whether statement X implied statement Y in the argument, or vice versa; whether a terminology introduced by the speaker is related to a very similar sounding terminology that you already knew about; and so forth). If you don't ask, you might be lost for the remainder of the talk; and usually speakers appreciate the feedback (it shows that at least one audience member is paying attention!) and the opportunity to explain things better, both to you and to the rest of the audience. However, questions which do not immediately enhance the flow of the talk are probably best left to after the end of the talk.

Besides the advice already on these web pages, the one thing I can offer you is that when you are learning by yourself, it becomes very important to find ways to really test your knowledge of the subject, since you do not have homework, exams, or other feedback available. Doing exercises from the textbook is of course one way to test yourself, though you should resist the temptation to "cheat", for instance by persuading yourself that you can do a problem without actually writing down all the details. But, as I already discuss in the above post, there are plenty of other usefully instructive tests you can make for yourself, for instance seeing whether you can somehow improve one of the lemmas in a text, or working through a special case of a theorem, etc.

Learn and relearn your field

Even fairly good students, when they have obtained the solution of the problem and written down neatly the argument, shut their books and look for something else. Doing so, they miss an important and instructive phase of the work. … A good teacher should understand and impress on his students the view that no problem whatever is completely exhausted.
One of the first and foremost duties of the teacher is not to give his students the impression that mathematical problems have little connection with each other, and no connection at all with anything

else. We have a natural opportunity to investigate the connections of a problem when looking back at its solution. (George Pólya, "How to Solve It")

Learning never really stops in this business, even in your chosen specialty; for instance I am still learning surprising things about basic harmonic analysis, more than ten years after writing my thesis in the topic.

Enjoy your work

No profit grows where is no pleasure ta'en;
In brief, sir, study what you most affect.
(William Shakespeare, "The Taming of the Shrew")

To really get anywhere in mathematics requires hard work. If you don't enjoy what you are doing, and do not derive satisfaction from your contributions, it will be difficult to put in the sustained amounts of energy required to succeed in the long term.

In general, it is much better to work in an area of mathematics which you enjoy, than one which you are working in simply because it is fashionable. For similar reasons, one should base one's work satisfaction on realistic achievements, such as advancing the state of knowledge in one's specialty, improving one's understanding of a field, or communicating this understanding successfully to others, rather than basing it on exceptionally rare events, such as spectacularly solving a major open problem, or achieving major recognition from one's peers. (Daydreams of glory may be pleasant to indulge in for a few moments, but they are poor sustenance for the patient, long-term effort required to actually make mathematical progress, and overly unrealistic expectations in this regard can lead to frustration.)

Enthusiasm can be infectious; one reason why you should attend talks and conferences is to find out what other exciting things are happening in your field (or in nearby fields), and to be reminded of the higher goals in your area (or in mathematics in general). A good talk can recharge your own interest in mathematics, and inspire your creativity.

Attend talks and conferences, even those not directly related to your work

Know how to listen, and you will profit even from those who talk badly. (Plutarch)

Modern mathematics is very much a collaborative activity rather than an individual one. You need to know what's going on elsewhere in mathematics, and what other mathematicians find interesting; this will often give valuable perspectives on your own work. This is true not just for talks in your immediate field, but also in nearby fields. (For much the same reason, I recommend studying at different places.) An inspiring talk can also increase your motivation in your own work and in the field of mathematics in general.

You also need to know who's who, both in your field and in neighboring ones, and to acquaint yourself with your colleagues. This way you will be much better prepared when it does turn out that your work has some new connections to other areas of mathematics, or when it becomes natural to work in collaboration with another mathematician. Talks and conferences are an excellent way to acquaint yourself with your mathematical community.

(Yes, it is possible to solve a major problem after working in isolation for years – but only *after* you first talk to other mathematicians and learn all the techniques, intuition, and other context necessary to crack such problems.)

Oh, and don’t expect to understand 100% of any given talk, especially if it is in a field you are not familiar with; as long as you learn *something*, the effort is not wasted, and the next time you go to a talk in that subject you will understand more. (One can always bring some of your own work to quietly work on once one is no longer getting much out of the talk.)

Advice on gifted education

If you can give your son or daughter only one gift, let it be enthusiasm. (Bruce Barton)

Education is a complex, multifaceted, and painstaking process, and being gifted does not make this less so. I would caution against any single “silver bullet” to educating a gifted child, whether it be a special school, private tutoring, home schooling, grade acceleration, or anything else; these are all options with advantages and disadvantages, and need to be weighed against the various requirements and preferences (both academic and non-academic) of the child, the parents, and the school. Since this varies so much from child to child, I cannot give any *specific* advice on a given child’s situation. [In particular, due to many existing time commitments and high volume of requests, I am unable to personally respond to any queries regarding gifted education.]

I can give a few *general* pieces of advice, though. Firstly, one should not focus overly much on a specific artificial benchmark, such as obtaining degree X from prestigious institution Y in only Z years, or on scoring A on test B at age C. In the long term, these feats will not be the most important or decisive moments in the child’s career; also, any short-term advantage one might gain in working excessively towards such benchmarks may be outweighed by the time and energy that such a goal takes away from other aspects of a child’s social, emotional, academic, physical, or intellectual development. Of course, one should still work hard, and participate in competitions if one wishes; but competitions and academic achievements should not be viewed as ends in themselves, but rather a way to develop one’s talents, experience, knowledge, and enjoyment of the subject.

Secondly, I feel that it is important to enjoy one’s work; this is what sustains and drives a person throughout the duration of his or her career, and holds burnout at bay. It would be a tragedy if a well-meaning parent, by pushing too hard (or too little) for the development of their child’s gifts in a subject, ended up accidentally extinguishing the child’s love for that subject. The pace of the child’s education should be driven more by the eagerness of the child than the eagerness of the parent.

Thirdly, one should praise one’s children for their efforts and achievements (which they can control), and not for their innate talents (which they cannot). This article by Po Bronson describes this point excellently. See also the Scientific American article “The secret to raising smart kids” for a similar viewpoint.

Finally, one should be flexible in one’s goals. A child may be initially gifted in field X, but decides that field Y is more enjoyable or is a better fit. This may be a better choice, even if Y is “less prestigious” than X; sometimes it is better to work in a less well known field that one feels competent and comfortable in, than in a “hot” but competitive field that one feels unsuitable for. (See also Ricardo’s law of comparative advantage.)

From Wikipedia:

Within the field of mathematics, Tao is known for his collaboration with British mathematician Ben J. Green of Oxford University; together they proved the Green–Tao theorem. Known for his collaborative mindset, by 2006, Tao had worked with over 30 others in his discoveries,[17] reaching 68 co-authors by October 2015.

In a book review, the British mathematician Timothy Gowers remarked on Tao's accomplishments:[18]

Tao's mathematical knowledge has an extraordinary combination of breadth and depth: he can write confidently and authoritatively on topics as diverse as partial differential equations, analytic number theory, the geometry of 3-manifolds, nonstandard analysis, group theory, model theory, quantum mechanics, probability, ergodic theory, combinatorics, harmonic analysis, image processing, functional analysis, and many others. Some of these are areas to which he has made fundamental contributions. Others are areas that he appears to understand at the deep intuitive level of an expert despite officially not working in those areas. How he does all this, as well as writing papers and books at a prodigious rate, is a complete mystery. It has been said that David Hilbert was the last person to know all of mathematics, but it is not easy to find gaps in Tao's knowledge, and if you do then you may well find that the gaps have been filled a year later.

Tao has won numerous mathematician honours and awards over the years.[19]

He is a Fellow of the Royal Society, the Australian Academy of Science (Corresponding Member), the National Academy of Sciences (Foreign member), the American Academy of Arts and Sciences, and the American Mathematical Society.[20] In 2006 he received the Fields Medal "for his contributions to partial differential equations, combinatorics, harmonic analysis and additive number theory", and was also awarded the MacArthur Fellowship. He has been featured in *The New York Times*, CNN, *USA Today*, *Popular Science*, and many other media outlets.[21]

As of 2019, Tao has published nearly 350 research papers and 18 books.[22] He has an Erdős number of 2.[23]

In 2018, Tao proved bounding for the de Bruijn–Newman constant. In 2019, Tao proved for the Collatz Conjecture using probability that almost all Collatz orbits attain almost bounded values.[24][25]